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Amortization System in the Simple Interest Regime: Comparison Between Three Proposals in the Case of Constant Amortization

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Abstract

This article compares three methodologies, two developed in Brazil and one in Italy, to amortize a loan using the constant amortization methods, in a simple capitalization regime. Due to the characteristics of this capitalization regime, two focal dates are studied, at the beginning and the end of the loan. The models are briefly described and a comparison between the models, two by two, is made, with the purpose of determining, from the point of view of a financial company, which one is the best methodology to be used.

Keywords: constant amortization methods, simple capitalization regime.

1. Introduction

Motivated by the concept of anatocism, which consists in applying interest upon interest, the Brazilian Judicial System, cf. *Jusbrasil* (2023), has repeatedly determined that financial contracts written in terms of compound interest should be substituted by contracts making use of simple interest. This has also occurred in the Italian Judicial System cf. Annibali et al. (2016).

However, a very recent Brazilian Law, Number 14.905, promulgated on June 28 of 2024, has established that what is defined as the legal rate, is the so called Selic rate, less the pertinent monetary correction. In other words, the legal rate is of compound interest. On the other hand, this new law does not preclude that financial contracts may be written specifying simple interest.

Disregarding the issue of the occurrence, or not, of anatocism in the compound interest regime, which is a controversial issue in Brazil, see Pucinni (2023) and De-Losso et al. (2020), as well in the Italian literature, see Annibali et al. (2016), our objective is to compare three distinct propositions, for the case of the constant amortization system, named in Brazil as "Sistema de Amortização Constante" (SAC) and in Italy "Piano de ammortamento Italiano", cf. Marcelli (2019), for contracts written in terms of simple interest.

In the case of adopting the compound interest regime, at the periodic rate $\boldsymbol{i}$, the specification of the date of equivalence comparison, between the value $\boldsymbol{F}$ and the sequence of $\boldsymbol{n}$ periodic payments, known as the focal date, cf. Ayres (1963), is not relevant. Because, whatever the focal date specified, the same condition of financial equivalence will always be satisfied between the financing amount and the sequence of periodic payments.

On the other hand, if rate $\boldsymbol{i}$ is of simple interest, the choice of the focal date is essential, because different focal dates lead to different results. In what follows, as they appear to be the most relevant, we will consider two focal dates that have been considered in the literature. The first, which appears to be the most natural, is the date on which the financing was granted, time 0. This, according to Mari and Aretusi (2018), is the one that should be considered. Furthermore, as observed in De-Losso et al. (2020), it is what follows from the provisions of paragraph 1 of article 15-B of Brazilian Act 4,380/64. The second is the date of the last payment, time $\boldsymbol{n}$. With the latter being proposed, in the case of constant payment, cf. Nogueira (2013), and in the case of constant amortization, see Rovina (2009) and Forger (2009). As well as in the case of the Italian literature, see Annibali et al. (2020).

We discuss three different methodologies. The first one was developed by Forger (2009) and extended by Lachtermacher and de Faro (2023b), a second one was developed by De-Losso and Santos (2023), both in Brazil. The third one was reported by Marcelli (2019) and Annibali et al. (2020), in Italy. All three methodologies use simple interest capitalization in the constant amortization method.

Our purpose here is to show which methodology a financial institution granting the loan will be better off by choosing one of them. As such, we will compare, considering the financial institution cost of capital, the present values of the corresponding sequences of interest parcels.

2. Forger's Methodology

Forger (2009) stipulates that the value F of the loan must be divided into two distinct components. One, called capitalizable and denoted as F^C, and the other, called non-capitalizable, being denoted as F^N. That is:

$$F = F^C + F^N \tag{1}$$

Denoting as S_k the outstanding balance at time k, immediately after the k^{th} payment P_k, and as A_k the amortization component at time k, which makes up the installment P_k, it is assumed that $S_k = S_k^C + S_k^N$, $P_k = P_k^C + P_k^N$, and $A_k = A_k^C + A_k^N$ for k = 1, 2, …, n. Furthermore, for k = 1, 2, …, n, where I_k denotes the interest component of the payment P_k, it is established that:

$$S_k^C = S_{k-1}^C - A_k^C = S_{k-1}^C - P_k^C \Leftrightarrow A_k^C = P_k^C \tag{2}$$

$$S_k^N = S_{k-1}^N - A_k^N = S_{k-1}^N + I_k - P_k^N \Leftrightarrow A_k^N = P_k^N - I_k \tag{3}$$

with the simple interest rate i being levied only on the capitalizable balance S_k^C. In other words, it is assumed that:

$$I_k = i \times S_{k-1}^C . \tag{4}$$

It should be noted that the interest component is not subdivided.

At time zero, when the loan is granted, the introduction of a weigh factor f is considered, with $0 \le f \le 1$, such that:

$$F = F^C + F^N \Leftrightarrow S_0 = S_0^C + S_0^N \tag{5}$$

with $S_0^C = F \times f$, $S_0^N = F \times (1-f)$ and $F = S_0$.

Additionally, as the capitalizable debt balance is supposed to decrease linearly, from its initial value $S_0^C = F \times f$, to the final value $S_n^C = 0$, and as, regardless of the particular amortization system that is adopted, it is established that $P^C = P_k^C = A_k^C = A^C$, whatever k, we have that:

$$P^C = A^C = (F/n) \times f . \tag{6}$$

Consequently, using the recursion given by (2), we obtain:

$$S_k^C = F \times f - k \times P^C = F \times f \times \left[(n-k)/n\right]. \tag{7}$$

Therefore, considering relationship (4), it follows that:

$$I_k = S_{k-1}^C \times i = F \times f \times i \times \left[(n-k+1)/n\right]. \tag{8}$$

In the case of the constant amortization system, we also have $A_k^N = A^N$ *for* $k = 1, 2, ..., n$.

Therefore, since $S_0^N = 0$, as the sum of the parcels of amortization must be equal to $F \times (1-f)$, we have:

$$A^N = \left[F \times (1-f)\right]/n . \tag{9}$$

Consequently, from equations (3) and (4), we obtain:

$$S_k^N = F \times (1-f) - k \times A^N = F \times (1-f) - k \times \left[F \times (1-f)\right]/n$$

or

$$S_k^N = F \times (1-f) \times (n-k)/n \tag{10}$$

and

$$P_k^N = A^N + I_k = \left[F \times (1-f)/n\right] + \left[F \times f \times i \times (n-k+1)/n\right]. \tag{11}$$

Until this point, we did not make any distinction between the focal dates. So, these formulas work for both cases. Forger (2009) established that the formula for the f factor, for the case where the focal date is at the end of the financing, is given by:

$$f = 1/\left[1 + 2 \times i \times (n-1)/3\right]. \tag{12}$$

Since the methodology developed by Lachtermacher and de Faro (2023b) can be used for every focal date, we will use it in the following examples.

2.1 Numerical Example

To compare the methodologies under scrutiny, we are going to use the same numerical example in all cases. A financing value of 100,000 units of capital, 12 periods, and a simple interest rate of 1% per period.

2.1.1 Forger Method - Focal Date at the Beginning of the Term (k = 0)

In this case, the financial equivalence, using simple interest capitalization, between the value F of the loan and the present value of the sequence of the n periodic payments, implies that:

$$F = \sum_{k=1}^{n} \frac{P_k}{1 + i \times k}. \tag{13}$$

Making use of the methodology developed in Lachtermacher and de Faro (2023b), the value of the weigh factor f, for the case of our numerical example, is 0.966126423. Furthermore, in the case of constant amortization, which value will be denoted as A, it follows:

$$A = A^C + A^N = \frac{F}{n} \times f + \frac{F \times (1 - f)}{n} = \frac{F}{n} \tag{14}$$

whose numerical value, in this case, is 8,333.33.

Table 1 shows the evolution of the debt in the case where the focal date is time 0 (that is, at the beginning of the term).

Table 1. Forger Method - Evolution of the Debt when the Focal Date is Time 0

Time (k)	I_k	A^N	A^C	P_k^N	P_k^C	S_k^N	S_k^C	S_k
0	---	---	---	---	---	3,387.36	96,612.64	100,000.00
1	966.13	282.28	8,051.05	1,248.41	8,051.05	3,105.08	88,561.59	91,666.67
2	885.62	282.28	8,051.05	1,167.90	8,051.05	2,822.80	80,510.54	83,333.33
3	805.11	282.28	8,051.05	1,087.39	8,051.05	2,540.52	72,459.48	75,000.00
4	724.59	282.28	8,051.05	1,006.87	8,051.05	2,258.24	64,408.43	66,666.67
5	644.08	28228	8,051.05	926.36	8,051.05	1,975.96	56,357.37	58,333.33
6	563.57	282.28	8,051.05	845.85	8,051.05	1,693.68	48,306.32	50,000.00
7	483.06	282.28	8,051.05	765.34	8,051.05	1,411.40	40,255.27	41,666.67
8	402.55	282.28	8,051.05	684.83	8,051.05	1,129.12	32,204.21	33,333.33
9	322.04	282.28	8,051.05	604.32	8,051.05	846.84	24,153.16	25,000.00
10	241.53	282.28	8,051.05	523.81	8,051.05	564.56	16,102.11	16,666.67
11	161.02	282.28	8,051.05	443.30	8,051.05	282.28	8,051.05	8,333.33
12	80.51	282.28	8,051.05	362.79	8,051.05	0.00	0.00	0.00
Σ	6,279.82	3,387.36	96,612,64	9,667.18	96,612.64	---	---	---

2.1.2 Forger Method - Focal Date at the End of the Term (k = n)

In this case, the financial equivalence, using simple interest capitalization, between the accumulated value of the loan and the accumulated value of the sequence of periodic payments, with the kth now being denoted by $\hat{P}_k$, implies that we must have:

$$F \times (1 + i \times n) = \sum_{k=1}^{n} \hat{P}_k \times \{1 + i \times (n - k)\}. \tag{15}$$

In the case of constant amortization, denoted by $\hat{A}$, it follows:

$$\hat{A} = A^C + \hat{A}^N = (F \times f'/n) + [F \times (1 - f')/n] = F/n. \tag{16}$$

Comparing expressions (14) and (16), we see that, although they have the same value, their components are different. As the weigh factor, now denoted as f', is also different. Making use of the methodology developed in Lachtermacher and de Faro (2023b), the value of f' is determined to be 0.931677019. Table 2 shows the evolution of the debt in this case.

Table 2. Forger Method - Evolution of the Debt when the Focal Date is Time *n*

Time(k)	$\hat{I}_k$	$\hat{A}_k^N$	$\hat{A}_k^C$	$\hat{P}_k^N$	$\hat{P}_k^C$	$\hat{S}_k^N$	$\hat{S}_k^C$	$\hat{S}_k$
0	---	---	---	---	---	6,832.30	93,167.70	100,000.00
1	931.68	569.36	7,763.98	1,501.04	7,763.98	6,262.94	85,403.73	91,666.67
2	854.04	569.36	7,763.98	1,423.40	7,763.98	5,693.58	77,639.75	83,333.33
3	776.40	569.36	7,763.98	1,345.76	7,763.98	5,124.22	69,875.78	75,000.00
4	698.76	569.36	7,763.98	1,268.12	7,763.98	4,554.87	62,111.80	66,666.67
5	621.12	569.36	7,763.98	1,190.48	7,763.98	3,985.51	54,347.83	58,333.33
6	543.48	569.36	7,763.98	1,112.84	7,763.98	3,416.15	46,583.85	50,000.00
7	465.84	569.36	7,763.98	1,035.20	7,763.98	2,846.79	38,819.88	41,666.67
8	388.20	569.36	7,763.98	957.56	7,763.98	2,277.43	31,055.90	33,333.33
9	310.56	569.36	7,763.98	879.92	7,763.98	1,708.07	23,291.93	25,000.00
10	232.92	569.36	7,763.98	802.28	7,763.98	1,138.72	15,527.95	16,666.67
11	155.28	569.36	7,763.98	724.64	7,763.98	569.36	7,763.98	8,333.33
12	77.64	569.36	7,763.98	647.00	7,763.98	0.00	0.00	0.00
Σ	6,055.90	6,832.30	93.167,70	12.888.20	93,167.70	---	---	---

3. De-Losso and Santos's Methodology - SACS

De-Losso and Santos (2023) developed an amortization system, whose acronym is SACS, that in Portuguese stands for "Sistema de Amortização em Capitalização Simples", meaning, Simple Capitalization Amortization System, which is very much analogous to the compound capitalization amortization system. Their model was built on the Multiple Contract System developed by De-Losso et al. (2013), which splits a contract of *n* payments into *n* contracts, each with a payment on date $k = 1, 2, \ldots, n$. With minor adjustments the proposed methodology can be applied to compound and simple capitalization.

3.1. SACS in Compound Interest Capitalization

SACS splits a contract of *n* payments into *n* contracts, each with a payment on date $k = 1, 2, \ldots, n$. And the corresponding principal value, $\overline{F}_k$, given by:

$$\overline{F}_k = \overline{P}_k / (1+i)^k \tag{17}$$

with the provision:

$$F = \sum_{k=1}^{n} \overline{F}_k \tag{18}$$

where $\overline{P}_k$ stands for the k^{th} payment.

De-Losso and Santos (2023) proposed that the interest component, now denoted as $\overline{I}_k$, of each installment, $\overline{P}_k$, is such that:

$$\overline{I}_k = i \times \overline{D}_k = i \times (1+i)^{k-1} \times \left(F - \sum_{\ell=1}^{k-1} \overline{F}_\ell\right) = i \times (1+i)^{k-1} \times \left(\sum_{\ell=k+1}^{n} \overline{F}_\ell\right) \tag{19}$$

where $\overline{D}_k$ denotes the remaining debt at time $k – 1$, which in this case is equal $\overline{S}_{k-1}$. And that $\overline{D}_1$ is equal to *F*.

Since the installment $\overline{P}_k$ is given by $\overline{P}_k = \overline{A}_k + \overline{I}_k$, where $\overline{A}_k$ is the amortization part of the installment *k,* which is supposed to be constant, it follows that:

$$\overline{A}_k = \overline{P}_k - \overline{I}_k \tag{20}$$

and the remaining debt $\overline{S}_k$, after paying the k^{th} installment, is given by:

$$\overline{S}_k = \overline{S}_{k-1} - \overline{A}_k = \overline{S}_{k-1} - \overline{P}_k + \overline{I}_k \tag{21}$$

3.1.1 Numerical Example

To compare the SACS methodology in compound and simple capitalization we are using the same example of the last section. That is, a financing value of 100.000 units of capital, with a term of 12 periods. However, now using the compound interest rate of 1% per period, and employing the constant amortization method.

The amortization of each period is constant and given, as well, by 8,333.33 units of capital. Table 3 shows the evolution of debt.

Table 3. De-Losso & Santos - Evolution of the Case with Compound Capitalization

Time(k)	$\bar{A}_k$	$\bar{D}_k$	$\bar{I}_k$	$\bar{P}_k$	$\bar{S}_k$
0	---	---	---	---	100,000.00
1	8,333.33	100,000.00	1,000.00	9,333.33	91,666.67
2	8,333.33	91,666.67	916.67	9,250.00	83,333.33
3	8,333.33	83,333.33	833.33	9,166.67	75,000.00
4	8,333.33	75,000.00	750.00	9,083.33	66,666.67
5	8,333.33	66,666.67	666.67	9,000.00	58,333.33
6	8,333.33	58,333.33	583.33	8,916.67	50,000.00
7	8,333.33	50,000.00	500.00	8,833.33	41,666.67
8	8,333.33	41,666.67	416.67	8,750.00	33,333.33
9	8,333.33	33,333.33	333.33	8,666.67	25,000.00
10	8,333.33	25,000.00	250.00	8,583.33	16,666.67
11	8,333.33	16,666.67	166.67	8,500.00	8,333.33
12	8,333.33	8,333.33	83.33	8,416.67	0.00
Σ	100,000.00	650,000.00	6,500.00	106,500.00	---

3.2 SACS in Simple Interest Capitalization – Focal Date Time Zero

Similarly to the case of compound interest, SACS splits a contract of *n* payments into *n* contracts, each with a payment on P'_k date $k = 1, 2, \ldots, n$. However, since we are assuming simple interest the principal value, F'_k, will be given by:

$$F'_k = \frac{P'_k}{(1 + i \times k)} \tag{22}$$

with

$$F = \sum_{k=1}^{n} F'_k \ . \tag{23}$$

De-Losso and Santos (2023) proposed that the interest component, I'_k, of each installment, P'_k, is such that:

$$I'_k = i \times D'_k = i \times \left(F - \sum\nolimits_{\ell=1}^{k-1} P'_\ell / (1 + i \times \ell) \right) = i \times \left(F - \sum\nolimits_{\ell=1}^{k-1} F'_\ell \right) = i \times \sum\nolimits_{\ell=k+1}^{n} F'_\ell \ . \tag{24}$$

Once more, it is interesting to note that D'_k is the remaining debt at time k – 1 and D'_1 is equal to F. It should be observed that the remaining debt at time k – 1 is not equal to S'_{k-1} as in the compound capitalization. Given the different feasibility properties between the two types of capitalization methods.

Similarly, the installment P'_k is given by $P'_k = A'_k + I'_k$, where A'_k is the amortization part of the installment k. That is:

$$A'_k = P'_k - I'_k \tag{25}$$

and the remaining debt S'_k, after paying the k^{th} installment, is given by:

$$S'_k = S'_{k-1} - A'_k = S'_{k-1} - P'_k + I'_k \tag{26}$$

3.2.1 Numerical Example

As a numerical application of the De-Losso and Santos (2023) methodology, we will consider the same example of section 2.1. In this case, the financial equivalence between the value ***F*** of the loan and the sequence of the ***n*** periodic payments, implies that:

$$F = \sum\nolimits_{k=1}^{n} P'_k / (1 + i \times k) \ . \tag{27}$$

Wherefore, since we are considering the case of constant amortization, $A' = F/n = 8{,}333.33$. Furthermore, observing that $S'_k = D'_{k-1},\ k = 0,1,\ldots,n,$ it is specified that:

$$D'_k = \sum_{\ell=k+1}^{n} \frac{P'_\ell}{(1+i\times\ell)} = F - \sum_{\ell=1}^{k} \frac{P'_{\ell-1}}{(1+i\times\ell)} \quad \text{and} \quad I'_k = D'_k \times i\ .$$

Table 4 shows the evolution of the debt in this case.

Table 4. SACS Evolution of the debt in the Case of the Focal Date at Time 0

Time (k)	A'_k	D'_k	I'_k	P'_k	S'_k
0	---	---	---	---	100,000.00
1	8,333.33	100,000.00	1,000.00	9,333.33	91,666.67
2	8,333.33	90,759.08	907.59	9,240.92	83,333.33
3	8,333.33	81,699.35	816.99	9,150.33	75,000.00
4	8,333.33	72,815.53	728.16	9,061.49	66,666.67
5	8,333.33	64,102.56	641.03	8,974.36	58,333.33
6	8,333.33	55,555.56	555.56	8,888.89	50,000.00
7	8,333.33	47,169.81	471.70	8,805.03	41,666.67
8	8,333.33	38,940.81	389.41	8,722.74	33,333.33
9	8,333.33	30,864.20	308.64	8,641.98	25,000.00
10	8,333.33	22,935.78	229.36	8,562.69	16,666.67
11	8,333.33	15,151.52	151.52	8,484.85	8,333.33
12	8,333.33	7,507.51	75.08	8,408.41	0.00
Σ	100,000.00	627,501.70	6,275.02	106,275.02	---

3.3 De-Losso and Santos's Method - Focal Date at the End of the Term (k = n)

In this case, the financial equivalence between the value ***F*** of the loan and the sequence of the n periodic payments, with the ***k***th now being denoted by $\hat{P}'_k$, implies that we have:

$$F \times (1+i\times n) = \sum_{k=1}^{n} \hat{P}'_k \times \left[1+i\times(n-k)\right] \ . \qquad (28)$$

SACS splits a contract of n payments into n contracts, each with the corresponding payment P'_k at date, for $k = 1, 2, \ldots, n$. The principal value, $\hat{F}'_k$, of the k^{th} subcontract, given by:

$$\hat{F}'_k = P'_k \times \left[1+i\times(n-k)\right] / (1+i\times k) \qquad (29)$$

with

$$F = \sum_{k=1}^{n} \hat{F}'_k \qquad (30)$$

Since $\hat{F}'_k$ is not, anymore, the present value of $\hat{P}'_k$, because we are now considering the focal date at time n, we must find an interest rate denoted as i_{eq}, that solves the following equation:

$$F = \sum_{k=1}^{n} \hat{P}'_k / (1+i_{eq}\times k)\ . \qquad (31)$$

Denoting

$$\hat{D}'_k = \sum_{\ell=k+1}^{n} P'_\ell / (1+i_{eq}\times\ell) = F - \sum_{\ell=1}^{k} \hat{P}'_{\ell-1} / (1+i_{eq}\times\ell) \qquad (32)$$

the interest component of the kth payment will be

$$\hat{I}'_k = D'_k \times i_{eq}\ . \qquad (33)$$

The numerical determination of i_{eq} may be accomplished, for instance, using Excel Goal Seek function. Analogously to the previous case, we will have $\hat{P}'_k = \hat{A}'_k + I'_k$, where $\hat{A}'_k$ is the amortization part of the installment k, so that:

$$\hat{A}'_k = \hat{P}'_k - I'_k \qquad (34)$$

and the remaining debt $\hat{S}'_k$, after paying the k^{th} installment, is given by:

$$\hat{S}'_k = S'_{k-1} - \hat{A}'_k = S'_{k-1} - \hat{P}'_k + I'_k\ . \qquad (35)$$

3.3.1 Numerical Example

Once more, as we are considering the case of constant amortization, we have $\hat{A}' = F / n = 8{,}333.33$. Table 5 shows the evolution of the debt in this case. In which the equivalent interest rate, i_{eq} numerically determined, is $\boldsymbol{i_{eq}}$=0.963173% p.p.

Table 5. SACS - Evolution of the Debt in the Case of the Focal Date at Time n

Time (k)	$\hat{A}'_k$	$\hat{D}'_k$	$\hat{I}'_k$	$\hat{P}'_k$	$\hat{S}'_k$
0	---	---	---	---	100,000.00
1	8,333.33	100,000.00	963.17	9,296.51	91,666.67
2	8,333.33	90,792.18	874.49	9,207.82	83,333.33
3	8,333.33	81,758.38	787.47	9,120.81	75,000.00
4	8,333.33	72,893.72	702.09	9,035.43	66,666.67
5	8,333.33	64,193.49	618.29	8,951.63	58,333.33
6	8,333.33	55,653.15	536.04	8,869.37	50,000.00
7	8,333.33	47,268.35	455.28	8,788.61	41,666.67
8	8,333.33	39,034.86	375.97	8,709.31	33,333.33
9	8,333.33	30,948.62	298.09	8,631.42	25,000.00
10	8,333.33	23,005.74	221.58	8,554.92	16,666.67
11	8,333.33	15,202.41	146.43	8,479.76	8,333.33
12	8,333.33	7,535.01	72.58	8,405.91	0.00
Σ	100,000.00	628,285.91	6,051.48	106,051.48	---

4. Italian Methodologies

In this case, we have two distinct methodologies which differ in terms of the focal date that is considered.

4.1 Italian Method - Focal Date at Time 0

Following Marcelli (2019), that focused attention only in the case of focal date at time 0, the interest calculus is based on the amortization quota paid until the time $\boldsymbol{k}$. Denoting as P''_k the k^{th} payment, the equivalence equation is given by:

$$F = \sum_{k=1}^{n} P''_k / \left(1 + k \times i\right). \tag{36}$$

In the case of a constant amortization, denoted by A'', we have:

$$A'' = F / n \tag{37}$$

which is the same as in the Forger's and De-Losso and Santos's methodologies. However, now, the interest calculus is given by:

$$I''_k = i \times \sum_{\ell=1}^{k} A''_\ell. \tag{38}$$

Table 6, considering the case of our numerical example, shows the evolution of the debt. Where S''_k denotes the outstanding debt at time k.

Table 6. Italian Method, Evolution of the Debt in the Case of the Focal Date at Time 0

Time (k)	I''_k	A''_k	P''_k	S''_k
0	---	---	---	100,000.00
1	83.33	8,333.33	8,416.67	91,666.67
2	166.67	8,333.33	8,500.00	83,333.33
3	250.00	8,333.33	8,583.33	75,000.00
4	333.33	8,333.33	8,666.67	66,666.67
5	416.67	8,333.33	8,750.00	58,333.33
6	500.00	8,333.33	8,833.33	50,000.00
7	583.33	8,333.33	8,916.67	41,666.67

8	666.67	8,333.33	9,000.00	33,333.33
9	750.00	8,333.33	9,083.33	25,000.00
10	833.33	8,333.33	9,166.67	16,666.67
11	916.67	8,333.33	9,250.00	8,333.33
12	1,000.00	8,333.33	9,333.33	0.00
Σ	6,500.00	100,000,00	106,500.00	---

Comparing the results respectively presented in Tables 1, 4 and 6, it should be observed that, effectively, the debt is paid off when the last payment is made. However, the total interest, in all cases, is not the same. But we must note that the total amount of interest in Table 6 is equal to the respective value of Table 3 (compound interest of SACS).

The second, which is crucial for the analysis that will be done in section 6, is that the sequence of differences between the corresponding interest sequences has a single signal variation when compared to the other methods.

4.2 Italian Method - Focal Date at the End of the Term (k = n)

On the other hand, the methodology is shown in Annibali et. al. (2020) was developed based on the focal date at the end of the financing, although not explicitly mentioned by the author. Denoting $\hat{P}_k''$ the k^{th} payment, the equivalence equation is given by:

$$F \times (1 + i \times n) = \sum_{k=1}^{n} \hat{P}'' \times \left[1 + i \times (n-k)\right]. \tag{39}$$

In the case of a constant amortization, denoted $\hat{A}''$, we have:

$$\hat{A}'' = F / n \quad . \tag{40}$$

Denoting by $\hat{I}_k''$ the interest component of $\hat{P}_k''$, it is established that:

$$\hat{I}_k'' = \hat{S}_{k-1}'' \times i / \left[1 + i \times (n-k)\right] \tag{41}$$

where $\hat{S}_k''$ denotes the outstanding debt at time *k*. Table 7 shows the evolution of the debt in the case of focal date at time *n*.

Table 7. Italian Method, Evolution of debt in the Case - Focal Date at Time *n*

Time (*k*)	$\hat{I}_k''$	$\hat{A}_k''$	$\hat{P}_k''$	$\hat{S}_k''$
0	---	---	---	100,000.00
1	900.90	8,333.33	9,234.23	91,666.67
2	833.33	8,333.33	9,166.67	83,333.33
3	764.53	8,333.33	9,097.86	75,000.00
4	694.44	8,333.33	9,027.78	66,666.67
5	623.05	8,333.33	8,956.39	58,333.33
6	550.31	8,333.33	8,883.65	50,000.00
7	476.19	8,333.33	8,809.52	41,666.67
8	400.64	8,333.33	8,733.97	33,333.33
9	323.62	8,333.33	8,656.96	25,000.00
10	245.10	8,333.33	8,578.43	16,666.67
11	165.02	8,333.33	8,498.35	8,333.33
12	83.33	8,333.33	8,416.67	0.00
Σ	6,060.48	100,000.00	106,060.48	---

Comparing the results respectively presented in Tables 2, 5 and 7, it should be observed that, effectively, the debt is paid off when the last payment is made. However, the total interest, in all cases, is not the same.

5. Checking the Financial Consistency of the Models

In de Faro (2014), focusing on the compound interest regime, it was established that an amortization system is financially consistent if the determination of the outstanding balance, or debt status, over the financing term, presents the same results according to each of the three classic procedures. In other words, the results arising from the application of the three classic procedures, the prospective, recurrence, and retrospective methods, must coincide.

Extending the concept of using financial consistency to the case of using the simple interest regime, Lachtermacher and Faro (2023a) showed that the three amortization systems proposed by Forger (2009), namely constant installment, constant amortization, and SACRE (increasing amortization system), are also financially consistent.

For the case under study, we will use the financial consistency verification methodology, developed in Lachtermacher and Faro (2023a), by calculating the outstanding balance for period ***k***=6 for all amortization systems. Rounding differences are expected to appear.

5.1 Forger's Method - Focal Date at Time 0

- Retrospective method

$$S_k = F - \sum_{\ell=1}^{k} A_\ell$$
$$S_6 = 100000 - \sum_{\ell=1}^{6} A_\ell = 100000 - A_1 - A_2 - \ldots - A_6 = 100000 - 6 \times A$$
$$S_6 = 120000 - 6 \times 8333.3333 = 50{,}000.00$$

- Prospective method

$$S_k = \sum_{\ell=k+1}^{n} \left(P_\ell - J_\ell\right)$$
$$S_6 = \begin{bmatrix} 8816.40 + 8735.89 + 8655.38 \\ +8574.86 + 8494.35 + 8413.84 \end{bmatrix} - \begin{bmatrix} 483.06 + 402.55 + 322.04 \\ +241.53 + 161.02 + 80.51 \end{bmatrix}$$
$$S_6 = 51690.72 - 1690.71 = 50{,}000.01$$

- Recurrence method

$$S_k = S_{k-1} + J_k - P_k \Rightarrow S_k = S_{k-2} + J_{k-1} - P_{k-1} + J_k - P_k, \text{ etc.}$$
$$S_k = S_0 + \sum_{\ell=1}^{k} J_\ell - \sum_{\ell=1}^{k} P_\ell$$
$$S_6 = S_0 + \sum_{\ell=1}^{6} J_\ell - \sum_{\ell=1}^{6} P_\ell = 100000 + 4589{,}10 - 54589{,}10 = 50{,}000.00$$

5.2 Forger's Method - Focal Date at Time n

- Retrospective method

$$\hat{S}_k = F - \sum_{\ell=1}^{k} A_\ell$$
$$\hat{S}_6 = 100000 - \sum_{\ell=1}^{6} A_\ell = 100000 - A_1 - A_2 - \ldots - A_6 = 100000 - 6 \times A$$
$$\hat{S}_6 = 100000 - 6 \times 8333.3333 = 50{,}000.00$$

- Prospective method

$$\hat{S}_k = \sum_{\ell=k+1}^{n} \left(\hat{P}_\ell - I_\ell\right) = \sum_{\ell=k+1}^{n} \left(\hat{P}_\ell\right) - \sum_{\ell=k+1}^{n} \left(I_\ell\right)$$
$$S_6 = \begin{bmatrix} 8799.17 + 8721.53 + 8643.89 \\ +8566.25 + 8488.61 + 8410.97 \end{bmatrix} - \begin{bmatrix} 465.84 + 388.20 + 310.56 \\ +232.92 + 155.28 + 77.64 \end{bmatrix}$$
$$S_6 = 51630.42 - 1630{,}44 = 49999.98$$

- Recurrence method

$$\hat{S}_k = S_{k-1} + \hat{I}_k - P_k \Rightarrow \hat{S}_k = S_{k-2} + \hat{I}_{k-1} - P_{k-1} + \hat{I}_k - P_k$$
$$\Rightarrow \hat{S}_k = S_0 + \sum_{\ell=1}^{k} \hat{I}_\ell - \sum_{\ell=1}^{k} P_\ell$$
$$\hat{S}_6 = S_0 + \sum_{\ell=1}^{6} \hat{I}_\ell - \sum_{\ell=1}^{6} P_\ell = 100000 + 4425{,}47 - 54425{,}47 = 50{,}000.00$$

5.3 De-Losso and Santos's Method – Focal Date at Time 0

- Retrospective method

$$S'_k = F - \sum_{\ell=1}^{k} A_\ell$$
$$S'_6 = 100000 - \sum_{\ell=1}^{6} A_\ell = 100000 - A_1 - A_2 - \ldots - A_6 = 100000 - 6 \times A$$
$$S'_6 = 120000 - 6 \times 8333.3333 = 50,000.00$$

- Prospective method

$$S'_k = \sum_{\ell=k+1}^{n} \left(P'_\ell - I'_\ell \right)$$
$$S'_6 = \begin{bmatrix} 8805.03 + 8722.74 + 8641.98 \\ +8562.69 + 8484.85 + 8408.41 \end{bmatrix} - \begin{bmatrix} 471.70 + 389.41 + 308.64 \\ +229.36 + 151.52 + 75.08 \end{bmatrix}$$
$$S'_6 = 51625.70 - 1625.71 = 49,999.99$$

- Recurrence method

$$S'_k = S'_{k-1} + I'_k - P'_k \Rightarrow S'_k = S'_{k-2} + I'_{k-1} - P'_{k-1} + I'_k - P'_k$$
$$\Rightarrow S'_k = S'_0 + \sum_{\ell=1}^{k} I'_\ell - \sum_{\ell=1}^{k} P'_\ell$$
$$S'_6 = S_0 + \sum_{\ell=1}^{6} I'_\ell - \sum_{\ell=1}^{6} P'_\ell = 100000 + 4649,32 - 54649.32 = 50,000.00$$

5.4 De-Losso and Santos's Method – Focal Date at Time n

- Retrospective method

$$\hat{S}'_k = F - \sum_{\ell=1}^{k} A_\ell$$
$$\hat{S}'_6 = 100000 - \sum_{\ell=1}^{6} A_\ell = 100000 - A_1 - A_2 - \ldots - A_6 = 100000 - 6 \times A$$
$$\hat{S}'_6 = 120000 - 6 \times 8333.3333 = 50,000.00$$

- Prospective method

$$\hat{S}'_k = \sum_{\ell=k+1}^{n} \left(\hat{P}'_\ell - I'_\ell \right)$$
$$\hat{S}'_6 = \begin{bmatrix} 8788.61 + 8709.31 + 8631.42 \\ +8554.92 + 8479.76 + 8405.91 \end{bmatrix} - \begin{bmatrix} 455.28 + 375.97 + 298.09 \\ +221.58 + 146.43 + 72.58 \end{bmatrix}$$
$$\hat{S}'_6 = 51569.93 - 1569.93 = 50,000.00$$

- Recurrence method

$$\hat{S}'_k = S'_{k-1} + \hat{I}'_k - P'_k \Rightarrow \hat{S}'_k = S'_{k-2} + \hat{I}'_{k-1} - P'_{k-1} + \hat{I}'_k - P'_k$$
$$\Rightarrow \hat{S}'_k = S'_0 + \sum_{\ell=1}^{k} \hat{I}'_\ell - \sum_{\ell=1}^{k} P'_\ell$$
$$\hat{S}'_6 = S'_0 + \sum_{\ell=1}^{6} \hat{I}'_\ell - \sum_{\ell=1}^{6} P'_\ell = 100000 + 4481.56 - 54481.56 = 50,000.00$$

5.5 Italian's Method - Focal Date at Time 0

- Retrospective method

$$S''_k = F - \sum_{\ell=1}^{k} A''_\ell$$
$$S''_6 = 100000 - \sum_{\ell=1}^{6} A''_\ell = 100000 - 6 \times A''$$
$$S''_6 = 100000 - 6 \times 8333.333 = 50,000.00$$

- Prospective method

$$S''_k = \sum_{\ell=k+1}^{n} \left(P''_\ell - I''_\ell\right) = \sum_{\ell=k+1}^{n} \left(P''_\ell\right) - \sum_{\ell=k+1}^{n} \left(I''_\ell\right)$$
$$S''_6 = \begin{bmatrix} 8916.67 + 9000.00 + 9083.33 \\ +9166.67 + 9250.00 + 9333.33 \end{bmatrix} - \begin{bmatrix} 583.33 + 666.67 + 750.00 \\ +833.33 + 9167.67 + 1000 \end{bmatrix}$$
$$S''_6 = 54,750.00 - 4,750.00 = 50,000.00$$

- Recurrence method

$$S''_k = S''_{k-1} + I''_k - P''_k \Rightarrow S''_k = S''_{k-2} + I''_{k-1} - P''_{k-1} + I''_k - P''_k$$
$$\Rightarrow S''_k = S''_0 + \sum_{\ell=1}^{k} I''_\ell - \sum_{\ell=1}^{k} P''_\ell$$
$$S''_6 = S''_0 + \sum_{\ell=1}^{6} I''_\ell - \sum_{\ell=1}^{6} P''_\ell = 100000 + 1750.00 - 51750.00 = 50,000.00$$

5.6 Italian's Method - Focal Date at Time n

- Retrospective method

$$\hat{S}''_k = F - \sum_{\ell=1}^{k} A''_\ell$$
$$\hat{S}''_6 = 100000 - \sum_{\ell=1}^{6} A''_\ell = 100000 - \hat{A}''_1 - A''_2 - \ldots - \hat{A}''_6 = 100000 - 6 \times A''$$
$$\hat{S}''_6 = 100000 - 6 \times 8333.3333 = 100000 - 50000.00 = 50,000.00$$

- Prospective method

$$\hat{S}''_k = \sum_{\ell=k+1}^{n} \left(\hat{P}''_\ell - I''_\ell\right) = \sum_{\ell=k+1}^{n} \left(\hat{P}''_\ell\right) - \sum_{\ell=k+1}^{n} \left(I''_\ell\right)$$
$$\hat{S}''_6 = \begin{bmatrix} 8788.61 + 8709.31 + 8631.43 \\ +8554.92 + 8479.76 + 8405.91 \end{bmatrix} - \begin{bmatrix} 455.28 + 375.97 + 298.09 \\ +221.58 + 146.43 + 72.58 \end{bmatrix}$$
$$\hat{S}''_6 = 51569.92 - 1569.92 = 50,000.00$$

- Recurrence method

$$\hat{S}''_k = S''_{k-1} + \hat{I}''_k - P''_k \Rightarrow \hat{S}''_k = S''_{k-2} + \hat{I}''_{k-1} - P''_{k-1} + \hat{I}''_k - P''_k$$
$$\Rightarrow \hat{S}''_k = S''_0 + \sum_{\ell=1}^{k} \hat{I}''_\ell - \sum_{\ell=1}^{k} P''_\ell$$
$$\hat{S}''_6 = S''_0 + \sum_{\ell=1}^{6} \hat{I}''_\ell - \sum_{\ell=1}^{6} P''_\ell = 100000 + 4481.56 - 54481.56 = 50,000.00$$

Therefore, we can conclude that all three methods are financially consistent.

6. Comparison of the Three Proposals

To determine the best option, from the point of view of the financial institution providing the loan, we will compare, for each focal date, the three procedures under analysis.

This will be done considering our numerical examples and taking into consideration the cost of capital of the financial institution. To compare the present values of the corresponding sequences of interest payments, according to the three methods being analysis.

6.1 Focal Date at Time 0

6.1.1 Comparing Forger (SAC) x De-Losso and Santos (SACS)

Table 8 presents the sequences of payments of interest, according to SAC (I_k) and SACS (I'_k), as well as their differences. Denoted as $d_k^1 = I_k - I'_k$; for $k = 1, 2, \ldots, n$.

Table 8. SAC x SACS – Focal Date at Time 0

Time (k)	I_k	I'_k	$d_k^1 = I_k - I'_k$
1	966.13	1,000.00	-33.87
2	885.62	907.59	-21.97
3	805.11	816.99	-11.89
4	724.59	728.16	-3.56
5	644.08	641.03	3.06
6	563.57	555.56	8.02
7	483.06	471.70	11.37
8	402.55	389.41	13.14
9	322.04	308.64	13.40
10	241.53	229.36	12.17
11	161.02	151.52	9.51
12	80.51	75.08	5.44
Σ	6,279.82	6,275.02	4.80

As depicted, the total interest payments are not equal and the sequence of differences has only one sign of change, indicating that the sequence of differences has a unique internal rate of return, as shown in de Faro (1974).

Figure 1 shows the interest sequences of the SAC and SACS methods for a 60-months term and focal date at time 0 and simple monthly interest rates of 0.5%, 1% and 2.0%. It is worth noting that the decrease in the values of the interest sequence of the SAC method is linear while that of the SACS method is exponential, with the initial values of the sequence of the SACS method being greater than those of the SAC method.

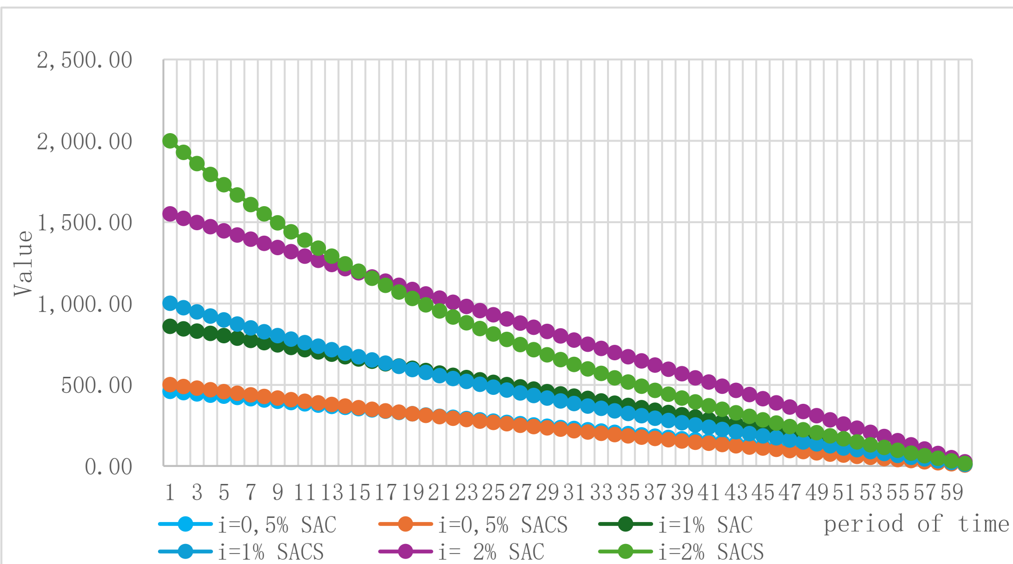

Figure 1. Interest Comparison – SAC vs. SACS, Focal Date at Time = 0, n = 60

Tables 9 and 10 compare the tax gain δ, between the SAC and SACS methods, given by the percentage difference of the present value of the interest sequence at rate ρ. Where ρ is the periodic rate that identifies the cost of capital of the financial institution.

Denoting

$$V_{SAC}^{0}(\rho)=\sum_{k=0}^{n} I_k \times (1+\rho)^{-k}$$

and

$$V_{SACS}^{0}(\rho)=\sum_{k=0}^{n} I'_k \times (1+\rho)^{-k}$$

we have:

$$\delta(\%)=\left[V_{SAC}^{0}(\rho)/V_{SACS}^{0}(\rho)-1\right]\times 100 \tag{42}$$

Table 9. Comparison Fiscal Gain δ – SAC x SACS, Focal Date at Time 0, i = 1% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	0.6359	0.0397	-0.5073	-1.0101	-1.4732	-1.9007
10	1.5284	-0.3579	-1.9986	-3.4282	-4.6774	-5.7731
15	2.2151	-1.2519	-4.0941	-6.4332	-8.3713	-9.9905
20	2.6290	-2.4973	-6.4522	-9.5327	-11.9662	-13.9192
25	2.7896	-3.9555	-8.8586	-12.4912	-15.2486	-17.3942
30	2.7355	-5.5225	-11.1949	-15.2177	-18.1751	-20.4242

Table 10. Comparison Fiscal Gain δ – SAC x SACS, Focal Date at Time 0, i = 2% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	2.5971	1.5754	0.6413	-0.2147	-1.0011	-1.7252
10	5.7822	2.7523	0.1381	-2.1248	-4.0920	-5.8102
15	8.3108	2.9448	-1.4033	-4.9518	-7.8746	-10.3072
20	10.1733	2.4316	-3.4588	-8.0064	-11.5802	-14.4407
25	11.4804	1.4711	-5.6962	-10.9625	-14.9444	-18.0391
30	12.3425	0.2477	-7.9339	-13.6944	-17.9188	-21.1320

Where ρ_a denotes the opportunity cost in annual terms. As we can see, there is no predominance of one method over the other, since the results present positive and negative values. Positive values mean that the SACS method presents a lower present value of the interest sequence, that is, a lower payment of fees. While negative values mean the opposite.

6.1.2 Comparing Forger (SAC) x Marcelli (Italian)

Table 11 presents the sequences of interest of SAC (I_k) and Italian (I''_k) for our numerical example, and their difference, when the focal date is at time 0. As can be seen the difference of the sequences change of sign just once and present different total values.

Table 11. SAC x Italian – Focal Date at Time 0

Time (k)	I_k	I''_k	$d_k^2 = I_k - I''_k$
1	966.13	83.33	882.79
2	885.62	166.67	718.95
3	805.11	250.00	555.11
4	724.59	333.33	391.26
5	644.08	416.67	227.42
6	563.57	500.00	63.57
7	483.06	583.33	-100.27
8	402.55	666.67	-264.11
9	322.04	750.00	-427.96

10	241.53	833.33	-591.80
11	161.02	916.67	-755.65
12	80.51	1,000.00	-919.49
Σ	6,279.82	6,500.00	-220.18

Figure 2 shows the interest sequences of the SAC and Italian methods for a 60-month period and focal date at time 0 and simple monthly interest rates of 0.5%, 1% and 2.0%. It is worth noting that the decrease in the values of the interest sequence of the SAC method is linear while the values increase linearly in the Italian method.

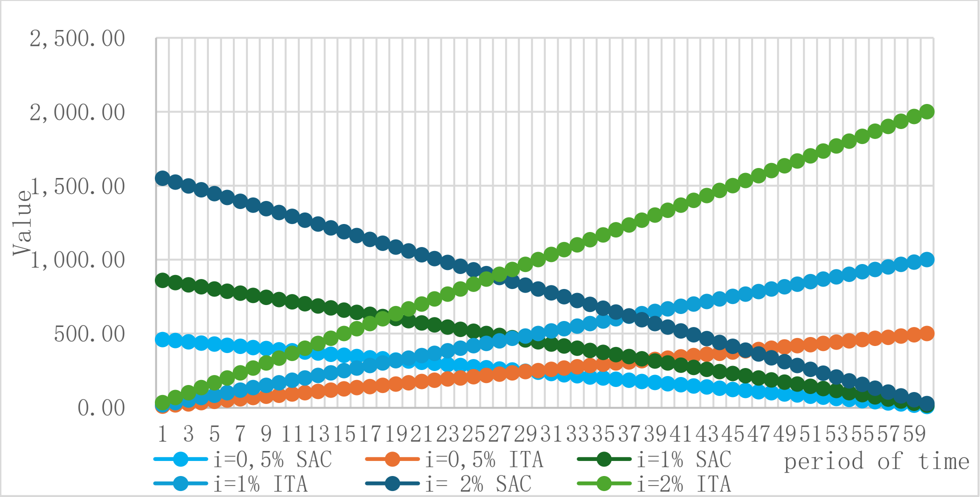

Figure 2. Interest Comparison – SAC vs. Italian, Focal Date at Time=0, n=60

Tables 12 and 13 compare the tax gain, δ' , between the SAC and Italian methods given by the percentage difference of the present value of the interest sequence at rate ρ, the opportunity cost rate.

Denoting

$$V_{SAC}^{0}(\rho)=\sum_{k=0}^{n} I_k \times (1+\rho)^{-k}$$

and

$$V_{ITA}^{0}(\rho)=\sum_{k=0}^{n} I_k'' \times (1+\rho)^{-k}$$

we have:

$$\delta'(\%)=\left[V_{SAC}^{0}(\rho)/V_{ITA}^{0}(\rho)-1\right]\times 100 \tag{43}$$

Table 12. Comparison Fiscal Gain δ' – SAC x Italian, Focal Date at Time 0, i = 1% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-7.0639	0.2750	7.7898	15.4565	23.2522	31.1548
10	-9.2929	5.5894	21.7499	39.0388	57.2845	76.3059
15	-9.1491	13.7948	39.8727	68.5810	99.3084	131.4256
20	-7.5950	24.0661	61.2262	102.5768	146.6251	192.0182
25	-5.0763	35.9711	85.0756	139.5057	196.5398	254.1326
30	-1.8329	49.2134	110.7310	177.9690	246.9652	315.3373

Table 13. Comparison Fiscal Gain δ' – SAC x Italian, Focal Date at Time 0, i = 2% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-16.0981	-9.4725	-2.6882	4.2333	11.2711	18.4055

10	-21.0836	-8.1358	5.9241	20.9656	36.8397	53.3885
15	-22.3660	-2.7599	19.5242	44.0561	70.3133	97.7581
20	-21.8314	4.9518	36.3868	71.3667	108.6287	147.0283
25	-20.1930	14.3176	55.6022	101.3642	149.3156	197.7366
30	-17.7889	24.9603	76.4789	132.7881	190.5697	247.8287

As we can see, there is no predominance of one method over the other, since the results present positive and negative values. Positive values mean that the Italian method presents a lower present value of the interest sequence, that is, a lower payment of fees, while negative values mean the opposite.

6.1.3 De-Losso (SACS) x Marcelli (Italian)

Table 14 presents the sequences of interest of SACS (I'_k) and Italian (I''_k) for our numerical example and their difference, for the focal date at time 0. As can be seen the difference of the sequences change of sign just once and present different total values.

Table 14. SACS x Italian – Focal Date at Time 0

Time (k)	I'_k	I''_k	$d^3_k = I'_k - I''_k$
1	1,000.00	83.33	916.67
2	907.59	166.67	740.92
3	816.99	250.00	566.99
4	728.16	333.33	394.82
5	641.03	416.67	224.36
6	555.56	500.00	55.56
7	471.70	583.33	-111.64
8	389.41	666.67	-277.26
9	308.64	750.00	-441.36
10	229.36	833.33	-603.98
11	151.52	916.67	-765.15
12	75.08	1,000.00	-924.92
Σ	6,275.02	6,500.00	-224.98

Figure 3 shows the interest sequences of the SACS and Italian methods for a 60-month period and focal date at time 0 and simple monthly interest rates of 0.5%, 1% and 2.0%. It is worth noting that the decrease in the values of the interest sequence of the SACS method is exponential while the values increase linearly in the Italian method.

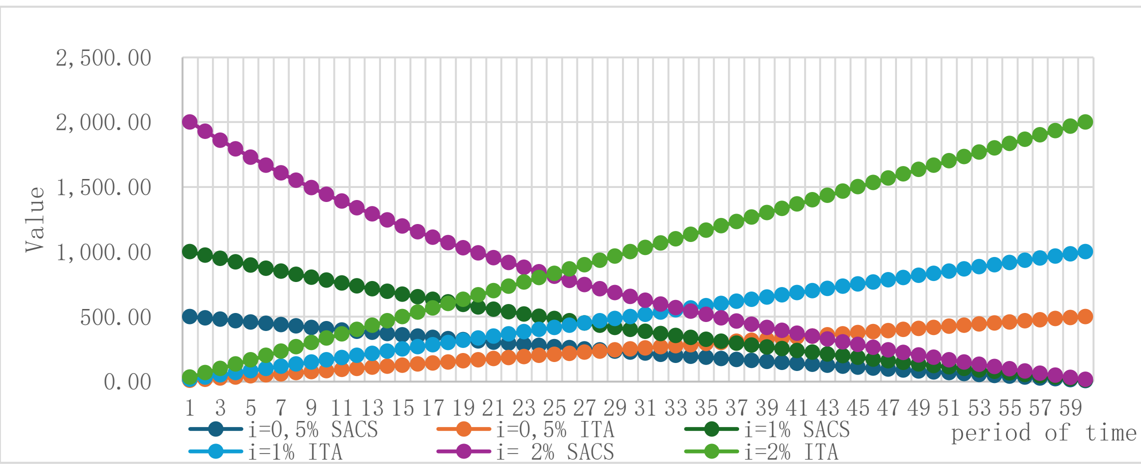

Figure 3. Interest Comparison – SACS vs. Italian Focal Date at Time=0, n=60

Tables 15 and 16 compare the tax gain, δ”, between the SACS and Italian methods given by the percentage difference of the present value of the interest sequence at rate ρ, the opportunity cost rate.

Denoting

$$V_{SACS}^{0}(\rho)=\sum_{k=0}^{n} I'_k \times (1+\rho)^{-k}$$

and

$$V_{ITA}^{0}(\rho)=\sum_{k=0}^{n} I''_k \times (1+\rho)^{-k}$$

we have:

$$\delta''(\%)=\left[V_{SACS}^{0}(\rho)/V_{ITA}^{0}(\rho)-1\right]\times 100 \tag{44}$$

Table 15. Comparison Fiscal Gain δ'' – SACS x Italian, Focal Date at Time 0, i=1% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-7.6511	0.2353	8.3394	16.6347	25.0952	33.6960
10	-10.6584	5.9687	24.2329	43.9745	65.0024	87.1078
15	-11.1179	15.2375	45.8437	80.1719	117.5175	157.1126
20	-9.9621	27.2438	72.3464	123.9228	180.1481	239.2372
25	-7.6524	41.5710	103.0643	173.6934	249.8939	328.7019
30	-4.4468	57.9353	137.2962	227.8621	324.0337	421.9393

Table 16. Comparison Fiscal Gain δ'' – SACS x Italian, Focal Date at Time 0, i=2% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-18.2219	-10.8766	-3.3083	4.4576	12.3963	20.4841
10	-25.3973	-10.5964	5.7780	23.5917	42.6780	62.8505
15	-28.3229	-5.5416	21.2254	51.5610	84.8712	120.4838
20	-29.0494	2.4604	41.2731	86.2811	135.9523	188.7216
25	-28.4116	12.6602	65.0009	126.1567	193.1209	263.2666
30	-26.8210	24.6516	91.6872	169.7253	254.0029	341.0261

As we can see, there is no predominance of one method over the other, since the results present positive and negative values. Positive values mean that the Italian method presents a lower present value of the interest sequence, that is, a lower payment of fees, while negative values mean the opposite

6.2 Focal Date at Time n

6.2.1 Comparing Forger (SAC) x De-Losso and Santos (SACS)

Table 17 presents the sequences of interest of SAC ($\hat{I}_k$) and SACS ($\hat{I}'_k$) for our numerical example and their difference, for the focal date at time n. As can be seen the difference of the sequences change of sign just once and present different total values.

Table 17. SAC x SACS – Focal Date at Time n

Time (k)	$\hat{I}_k$	$\hat{I}'_k$	$\hat{d}_k^1=\hat{I}_k-\hat{I}'_k$
1	931.68	963.17	-31.50
2	854.04	874.49	-20.45
3	776.40	787.47	-11.08
4	698.76	702.09	-3.33
5	621.12	618.29	2.82

6	543.48	536.04	7.44
7	465.84	455.28	10.56
8	388.20	375.97	12.23
9	310.56	298.09	12.47
10	232.92	221.58	11.33
11	155.28	146.43	8.85
12	77.64	72.58	5.06
Σ	6,055.90	6,051.48	4.42

Figure 4 shows the interest sequences of the SAC and SACS methods for a 60-month period and focal date at time n and simple monthly interest rates of 0.5%, 1% and 2.0%. It is worth noting that the decrease in the values of the interest sequence of the SAC method is linear while the values increase exponentially in the SACS method, with the initial values of the sequence of the SACS method being greater than those of the SAC method.

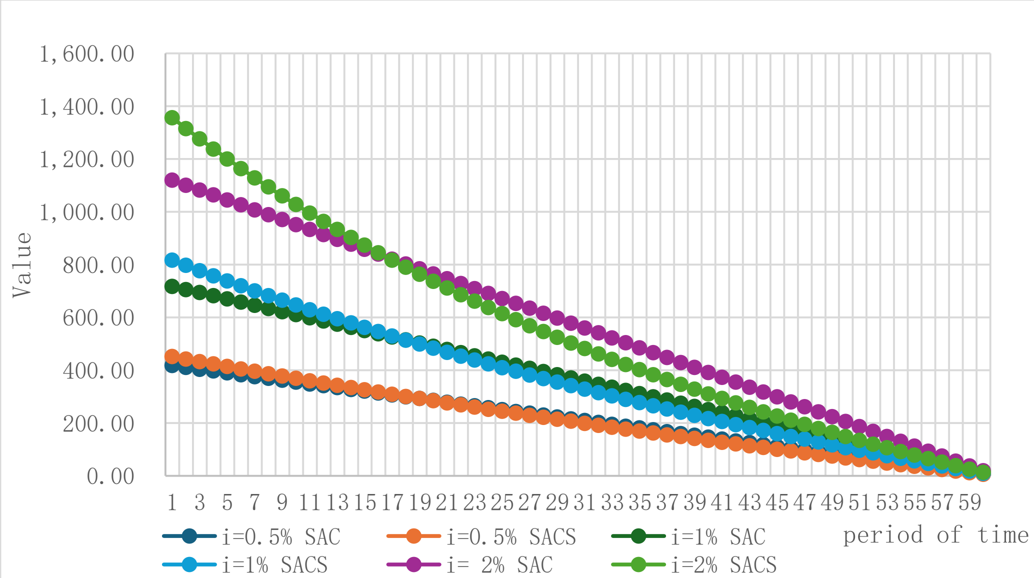

Figure 4. Interest Comparison – SAC vs. SACS, Focal Date at Time n, n = 60

Tables 18 and 19 compare the tax gain, $\hat{\delta}$, between the SAC and SACS methods given by the percentage difference of the present value of the interest sequence at rate ρ, the opportunity cost rate.

Denoting

$$V_{SAC}^{n}(\rho)=\sum_{k=0}^{n}\hat{I}_k\times(1+\rho)^{-k}$$

and

$$V_{SACS}^{n}(\rho)=\sum_{k=0}^{n}\hat{I}'_k\times(1+\rho)^{-k}$$

we have:

$$\hat{\delta}(\%)=\left[V_{SAC}^{n}(\rho)/V_{SACS}^{n}(\rho)-1\right]\times 100 \tag{45}$$

Table 18. Comparison Fiscal Gain $\hat{\delta}$ – SAC x SACS, Focal Date Time n, i = 1% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	0.4306	-0.0739	-0.5370	-0.9631	-1.3557	-1.7184
10	0.6432	-0.7674	-1.9986	-3.0745	-4.0168	-4.8446

15	0.4522	-1.9065	-3.8545	-5.4661	-6.8059	-7.9278
20	-0.0143	-3.2488	-5.7720	-7.7506	-9.3189	-10.5791
25	-0.6494	-4.6490	-7.5976	-9.7971	-11.4702	-12.7712
30	-1.3826	-6.0285	-9.2709	-11.5836	-13.2836	-14.5726

Table 19. Comparison Fiscal Gain $\hat{\delta}$ – SAC x SACS –Focal Date at Time *n*, *i*=2% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	1.4570	0.6948	-0.0035	-0.6447	-1.2346	-1.7787
10	1.9733	0.0838	-1.5598	-2.9918	-4.2433	-5.3410
15	1.7107	-1.2448	-3.6761	-5.6820	-7.3468	-8.7391
20	1.0769	-2.8074	-5.8263	-8.1882	-10.0584	-11.5607
25	0.2645	-4.3999	-7.8271	-10.3798	-12.3209	-13.8308
30	-0.6309	-5.9341	-9.6251	-12.2555	-14.1897	-15.6574

As we can see, there is no predominance of one method over the other, since the results present positive and negative values. Positive values mean that the SACS method presents a lower present value of the interest sequence, that is, a lower payment of fees, while negative values mean the opposite.

6.2.2 Forger (SAC) x Italian (Annibali et al.)

Table 20 presents the sequences of interest of SAC ($\hat{I}_k$) and Italian ($\hat{I}''_k$) for our numerical example and their difference, for the focal date at time *n*. As can be seen the difference of the sequences change of sign just once and present different total values.

Table 20. SAC x Italian, Focal Date at Time *n*

Time (*k*)	$\hat{I}_k$	$\hat{I}''_k$	$\hat{d}_k^2 = \hat{I}_k - I''_k$
1	931.68	900.90	30.78
2	854.04	833.33	20.70
3	776.40	764.53	11.87
4	698.76	694.44	4.31
5	621.12	623.05	-1.93
6	543.48	550.31	-6.84
7	465.84	476.19	-10.35
8	388.20	400.64	-12.44
9	310.56	323.62	-13.07
10	232.92	245.10	-12.18
11	155.28	165.02	-9.74
12	77.64	83.33	-5.69
Σ	6,055.90	6,060.48	-4.58

Figure 5 shows the interest sequences of the SAC and Italian methods for a 60-month period and focal date at time *n* and simple monthly interest rates of 0.5%, 1% and 2.0%. It is worth noting that the decrease in the values of the interest sequence of the SAC method is linear while the Italian method is decreasing nonlinearly.

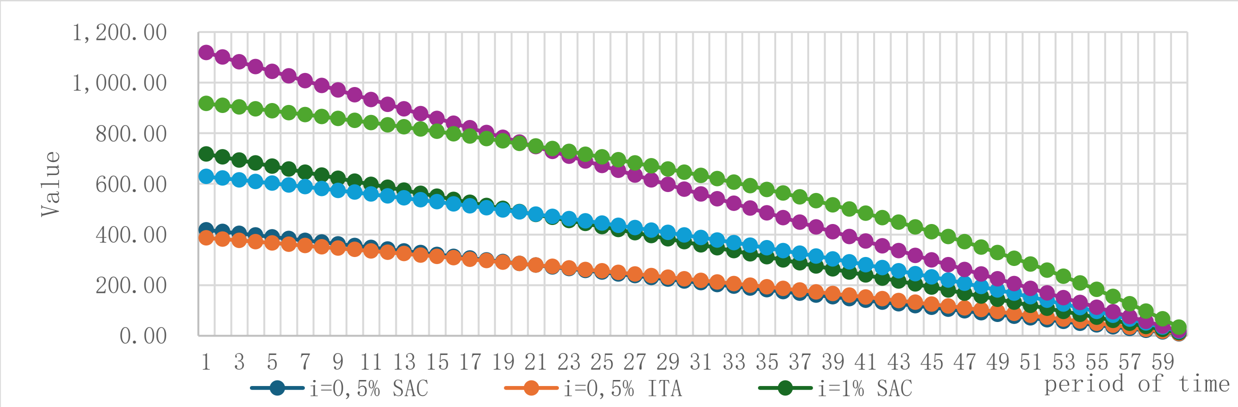

Figure 5. Interest Comparison – SAC vs. Italian Focal Date at Time n, n = 60

Tables 21 and 22 compare the tax gain, $\hat{\delta}'$, between the SAC and Italian methods given by the percentage difference of the present value of the interest sequence at rate ρ, the opportunity cost rate.

Defining

$$V_{SAC}^{n}(\rho)=\sum_{k=0}^{n}\hat{I}_k \times (1+\rho)^{-k}$$

and

$$V_{ITA}^{n}(\rho)=\sum_{k=0}^{n}\hat{I}_k'' \times (1+\rho)^{-k}$$

we have:

$$\hat{\delta}'(\%)=\left[V_{SAC}^{n}(\rho)/V_{ITA}^{n}(\rho)-1\right]\times 100 \tag{46}$$

Table 21. Comparison Fiscal Gain $\hat{\delta}'$ – SAC x Italian, Focal Date at Time n, i = 1%p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-0,503978	0,071587	0,604713	1,0989666	1,5576635	1,9838664
10	-0,883032	0,88759	2,469068	3,8759051	5,1247604	6,2327644
15	-0,798909	2,39852	5,130685	7,4403712	9,3835068	11,017607
20	-0,315436	4,358361	8,149912	11,174879	13,57766	15,493612
25	0,469863	6,566166	11,23211	14,734161	17,365053	19,366715
30	1,474698	8,876935	14,19899	17,958713	20,645386	22,61376

Table 22. Comparison Fiscal Gain $\hat{\delta}'$ – SAC x Italian–Focal Date Time n, i = 2% p.m. (per month)

	ρ_a(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-1,8901	-0,9515	-0,0795	0,7310	1,4848	2,1867
10	-3,0712	-0,4604	1,8851	3,9807	5,8465	7,5051
15	-3,1869	1,2557	5,0774	8,3193	11,0491	13,3426
20	-2,6229	3,6162	8,7063	12,7694	15,9878	18,5420
25	-1,6309	6,2704	12,3365	16,8741	20,2602	22,8166
30	-0,3661	9,0069	15,7429	20,4664	23,8091	26,2354

As we can see, there is no predominance of one method over the other, since the results present positive and negative values. Positive values mean that the Italian method presents a lower present value of the interest sequence, that is, a lower payment of fees, while negative values mean the opposite.

6.2.3- De-Losso (SACS) x Italian (Annibali et al.)

Table 23 presents the sequences of interest of SACS ($\hat{I}'_k$) and Italian ($\hat{I}''_k$) for our numerical example and their difference, for the focal date at time *n*. As can be seen the difference of the sequences change of sign just once and present different total values.

Table 23. SACS x Italian – Focal Date at Time *n*

Time (*k*)	$\hat{I}'_k$	$\hat{I}''_k$	$\hat{d}^3_k = \hat{I}'_k - I''_k$
1	963,17	900,90	62,27
2	874,49	833,33	41,15
3	787,47	764,53	22,95
4	702,09	694,44	7,65
5	618,29	623,05	-4,76
6	536,04	550,31	-14,28
7	455,28	476,19	-20,91
8	375,97	400,64	-24,67
9	298,09	323,62	-25,54
10	221,58	245,10	-23,51
11	146,43	165,02	-18,59
12	72,58	83,33	-10,76
Σ	6.051,48	6.060,48	-9,00

Figure 6 shows the interest sequences of the SACS and Italian methods for a 60-month period and focal date at time *n* and simple monthly interest rates of 0.5%, 1% and 2.0%. It is worth noting that decrease in the values of the interest sequence of both methods are nonlinear but in a different way, while SACS is convex, the Italian method presents a concave shape.

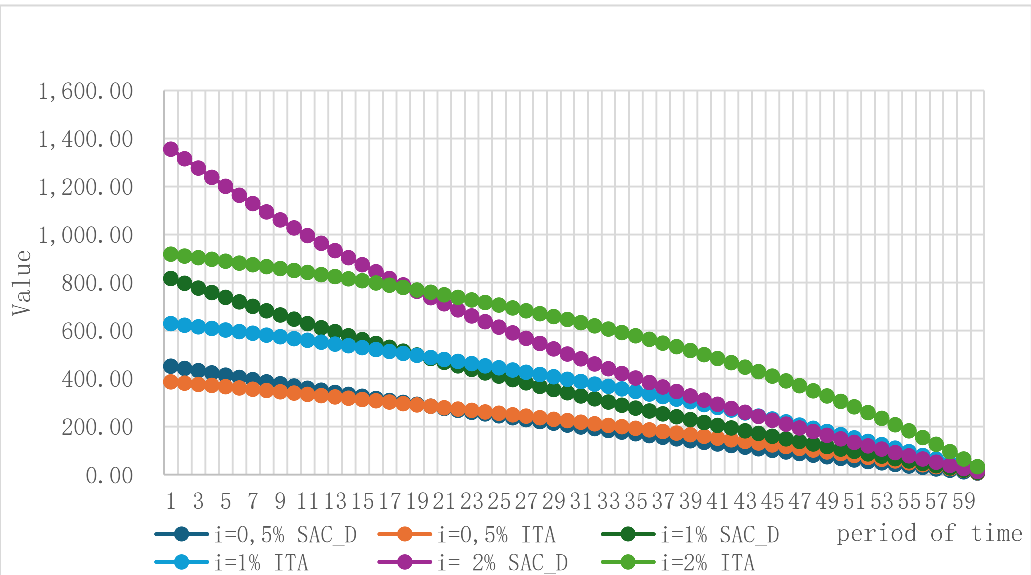

Figure 6. Interest Comparison – SAC vs. Italian, Focal Date at Time *n*, *n* = 60

Tables 24 and 25 compare the tax gain, $\hat{\delta}''$, between the SACS and Italian methods given by the percentage difference of the present value of the interest sequence at rate ρ, the opportunity cost rate.

Denoting

$$V_{SACS}^{n}(\rho) = \sum_{k=0}^{n} \hat{I}'_k \times (1+\rho)^{-k}$$

and

$$V_{ITA}^{n}(\rho) = \sum_{k=0}^{n} \hat{I}''_k \times (1+\rho)^{-k}$$

we have:

$$\hat{\delta}''(\%) = \left[V_{SACS}^{n}(\rho)/V_{ITA}^{n}(\rho) - 1\right] \times 100 \quad (47)$$

Table 24. Comparison Fiscal Gain $\hat{\delta}''$ – SACS x Italian, Focal Date at Time *n*, *i*=1% p.m. (per month)

	ρa(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-0,9306	0,1456	1,1479	2,0821	2,9534	3,7670
10	-1,5164	1,6678	4,5588	7,1708	9,5241	11,6414
15	-1,2455	4,3887	9,3454	13,6527	17,3718	20,5767
20	-0,3012	7,8625	14,7748	20,5155	25,2495	29,1573
25	1,1266	11,7620	20,3779	27,1956	32,5712	36,8432
30	2,8974	15,8616	25,8681	33,4127	39,1264	43,5298

Table 25. Comparison Fiscal Gain $\hat{\delta}''$ – SACS x Italian, Focal Date at Time *n*, *i*=2% p.m. (per month)

	ρa(%)					
n(years)	5%	10%	15%	20%	25%	30%
5	-3,2990	-1,6349	-0,0760	1,3846	2,7535	4,0372
10	-4,9469	-0,5437	3,4995	7,1876	10,5370	13,5709
15	-4,8152	2,5320	9,0876	14,8448	19,8546	24,1963
20	-3,6604	6,6092	15,4317	22,8267	28,9590	34,0378
25	-1,8904	11,1614	21,8758	30,4104	37,1595	42,5296
30	0,2665	15,8835	28,0698	37,2923	44,2824	49,6699

As we can see, there is no predominance of one method over the other, since the results present positive and negative values. Positive values mean that the Italian method presents a lower present value of the interest sequence, that is, a lower payment of fees, while negative values mean the opposite

7. Conclusion

In this article, we have compared three different systems with constant amortization in simple capitalization regime, using two focal dates, at the beginning and end of the financing. These methods were developed by Forger (2009) extended by Lachtermacher and de Faro(2023b), De-Losso and Santos (2023), in Brazil, and in Italy, by Annibali et al. (2020) and Marcelli (2019). From the point of view of the financing company, none of the methods were superior to the others. In all the analyses performed, the result varies with the financing company's cost of capital, the simple interest rate of the contract and the financing period. Therefore, we must conduct a simulation with the three methods to determine the best option in each case, for the financing company.

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